An Incremental Complementary Energy Method of Nonlinear Stress Analysis

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A stress analysis capability to treat orthotropic physically nonlinear materials that behave differently in tension and compression is presented. The analysis is an energy search method based on an incremental complementary energy formulation. Load incrementation is used. Stress solutions are obtained by formulating and minimizing the incremental complementary energy associated with each load increment. Material behavior is conservative during each load increment. Structures are idealized as an assemblage of discrete elements. An Airy stress function is approximated by a sum of unknown coefficients times interpolation polynomials for each element. Equilibrium between adjacent elements is satisfied identically by imposing equality constraints on the unknown coefficients. Applications to structural elements loaded in plane stress or plane strain are presented. Correlations with some experimental results available in the literature are made.

Introduction

RECENT advances in materials science have broadened the range of available structural materials to include several new composites. While exhibiting inviting qualities, the composite materials possess, to varying degrees, orthotropic and temperature dependent physical properties. The stress-strain behavior is often nonlinear and different in tension than compression. The effective structural use of these materials requires analysis methods capable of coping with their response characteristics.

The analysis presented here constitutes an effort to improve the prediction of structural behavior by treating material nonlinearities, orthotropicity and different behavior in tension than compression. Illustrative applications to simple structural components loaded in plane stress or plane strain are given.

At the outset, a description of the incremental complementary energy approach is presented in terms of a plane stress thermal problem with an isotropic linear strain hardening material that behaves the same in tension and compression. The stress solution obtained, using a discrete element implementation of the incremental complementary energy method, is shown to be in substantial agreement with an established reference solution. Subsequently, the steps to include material orthotropicity and different behavior in tension and compression are presented. Applications of the extended analysis are made to some axisymmetric plane strain problems and correlation with some available experimental results is presented.

The complementary energy method has not been applied as extensively as the potential energy method to nonlinear structural analysis problems. Some applications to trusses stressed beyond the proportional limit but assuming small displacements, rotations, and strains are reported in Refs. 1 and 2. In Ref. 3, the complementary energy method is employed to deal with trusses including the influence of finite displacements. The complementary energy principle has been used to construct equilibrium models within the context of finite element methods. For example, in Ref. 4, linear finite element analysis is based upon assumed stress distributions that satisfy the equilibrium conditions.

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Analysis

The method of stress analysis reported herein is termed an incremental complementary energy approach because it is an adaptation of the complementary energy formulation to incremental loading. The method may be briefly described as follows. The load is divided into increments. As each successive load increment is applied to the structure, an incremental complementary energy is formulated and minimized to obtain the stress distribution. This stress distribution is used as an initial stress state and the load is incremented upward. This process is continued until the desired load level is reached. Mechanical or thermal loadings are "built up" in a quasi-static fashion using a time independent value for the load during each load increment.

Initially, to introduce the ideas involved, a description of the incremental complementary energy approach is presented in terms of an isotropic plane stress thermal problem with a linear strain hardening material that behaves the same in tension and compression.

Incremental Complementary Energy Density

The incremental complementary energy density is the change in complementary energy density, Δu_c, between stress distribution state $Q - 1(\sigma_{x,Q-1}, \sigma_{y,Q-1}, \tau_{xy,Q-1})$ and stress distribution state $Q(\sigma_{xQ},\sigma_{yQ},\tau_{xyQ})$. By definition the incremental complementary energy density Δu_c, for the case of plane stress, linear strain displacement relations, and small rotations is

$$\Delta u_c = \int_{\sigma_{x,Q-1}}^{\sigma_{xQ}} \epsilon_x d\sigma_x + \int_{\sigma_{y,Q-1}}^{\sigma_{yQ}} \epsilon_y d\sigma_y + \int_{\tau_{xy,Q-1}}^{\tau_{xyQ}} \gamma_{xy} d\tau_{xy} \tag{1}$$

Stress distribution state $Q - 1$ is associated with load level $Q - 1$ and is assumed to be known. Stress distribution state Q is associated with load level Q arrived at by incrementing the load level upward from load level $Q - 1$.

The mechanical and thermal loading sequence considered here is such that if a point in the material is in the strain

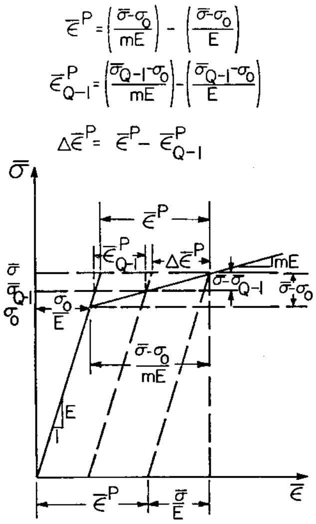

Fig. 1 Linear strain hardening.

hardening range, further thermal or mechanical loading does not decrease the value of the equivalent stress for that point. For points that have not reached the strain hardening range, additional thermal or mechanical load increments can increase, decrease, or not change the value of the equivalent stress.

Stress-Strain Temperature Relations

The stress-strain temperature relations for the Qth load increment, assuming the behavior in tension and compression to be the same, are taken to be of the following form:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \sum_{q=1}^{Q} (\alpha\Delta T)_q + \sum_{q=1}^{Q-1} \Delta\epsilon_x{}^{(q)P} + \Delta\epsilon_x{}^P \quad (2)$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} + \sum_{q=1}^{Q} (\alpha\Delta T)_q + \sum_{q=1}^{Q-1} \Delta\epsilon_y{}^{(q)P} + \Delta\epsilon_y{}^P \quad (3)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} + \sum_{q=1}^{Q-1} \Delta\gamma_{xy}{}^{(q)P} + \Delta\gamma_{xy}{}^P \quad (4)$$

In Eqs. (2) and (3), the first two terms are the elastic strains, the third term is the accumulated thermal strain up to and including the Qth load increment, the fourth term is the accumulated plastic strain from the first $Q - 1$ load increments, and the fifth term represents the incremental plastic strain due to the Qth load increment. In Eq. (4), the first term is the elastic shear strain, the second term is the accumulated plastic shear strain from the first $Q - 1$ load increments, and the third term is the incremental plastic shear strain due to the Qth load increment.

The incremental plastic strains are assumed to be given by the Prandtl-Reuss relations

$$\Delta\epsilon_x{}^P = (\sigma_x - \tfrac{1}{2}\sigma_y)(\Delta\bar{\epsilon}^P/\bar{\sigma}) \quad (5)$$

$$\Delta\epsilon_y{}^P = (\sigma_y - \tfrac{1}{2}\sigma_x)(\Delta\bar{\epsilon}^P/\bar{\sigma}) \quad (6)$$

$$\Delta\gamma_{xy}{}^P = 3\tau_{xy}(\Delta\bar{\epsilon}^P/\bar{\sigma}) \quad (7)$$

where $\Delta\bar{\epsilon}^P$ is the incremental equivalent plastic strain given by

$$\Delta\bar{\epsilon}^P = 2/(3)^{1/2}[(\Delta\epsilon_x{}^P)^2 + (\Delta\epsilon_y{}^P)^2 + \Delta\epsilon_x{}^P\Delta\epsilon_y{}^P + (\Delta\gamma_{xy}{}^P)^2]^{1/2} \quad (8)$$

and the equivalent stress, $\bar{\sigma}$, is defined by

$$\bar{\sigma} = (\sigma_x{}^2 - \sigma_x\sigma_y + \sigma_y{}^2 + 3\tau_{xy}{}^2)^{1/2} \quad (9)$$

A basic assumption of the incremental strain theory is that a value for the quantity $\Delta\bar{\epsilon}^P/\bar{\sigma}$ can be obtained from the uniaxial stress-strain curve of the material. This statement is equivalent to the assumption that there is a unique relationship between $\bar{\sigma}$ and $\bar{\epsilon}^P$.

For a linear strain hardening material (see Fig. 1)

$$\Delta\bar{\epsilon}^P/\bar{\sigma} = (1/E)[(1 - m)/m]\{1 - [\bar{\sigma}_{Q-1}/\bar{\sigma}]\} \quad (10)$$

where E is the modulus of elasticity, m is the linear strain hardening parameter, and $\bar{\sigma}_{Q-1}$ is the value of the equivalent stress at the previous load level.

Conditions to Insure Incrementally Conservative Behavior

It is important to establish that the material behavior between stress state $Q - 1$ and stress state Q (i.e., during any one load increment) is conservative. The material behavior is conservative if there is no net change in complementary energy as a result of going from stress state $Q - 1$ to stress state Q and then back to stress state $Q - 1$. Stated in equation form, the material behavior is conservative if

$$\int_{\sigma_{Q-1}}^{\sigma_Q} du_c + \int_{\sigma_Q}^{\sigma_{Q-1}} du_c = \oint du_c = 0 \quad (11)$$

where

$$du_c = \epsilon_x d\sigma_x + \epsilon_y d\sigma_y + \gamma_{xy} d\tau_{xy} \quad (12)$$

It can be shown that Eq. (11) holds if the following equations are satisfied:

$$\partial\gamma_{xy}/\partial\sigma_y = \partial\epsilon_y/\partial\tau_{xy},\ \partial\gamma_{xy}/\partial\sigma_x = \partial\epsilon_x/\partial\tau_{xy} \qquad \partial\epsilon_x/\partial\sigma_y = \partial\epsilon_y/\partial\sigma_x \quad (13)$$

Thus, material behavior is conservative if Eqs. (13) are satisfied. The stress-strain relations represented by Eqs. (2–4) satisfy Eq. (13). Therefore, the incremental complementary energy density [see Eq. (1)] is independent of the path taken to arrive at stress state Q from stress state $Q - 1$.

In order to further clarify the incremental conservative behavior described here, it is correct to say that during any one load increment, the model does not admit nonconservative behavior such as "elastic unloading" for $\bar{\sigma}$ greater than the $\bar{\sigma}$ at the end of the previous load increment. That is, the stress path shown in Fig. 2 by the sequence 1-2-3 is not admissible, by the model, during any single load increment. However, in addition to the theoretical verification for the method given in Appendix A, there is a theoretical verification for a specialized case of elastic unloading in the strain hardening range. For this special type of unloading it must be known a priori which points in the material will "unload" and these points are assumed to unload as described by the sequence 1–4 in Fig. 2, where the equivalent stress at the end of the previous load increment is at point 1 of the figure. In general, the problem of determining which points unload is a complex one, as can be seen from Ref. 5.

Description of Incremental Complementary Energy Approach

The incremental complementary energy density can be expressed as a function of the unknown stress state Q by substituting Eqs. (2–4) into Eq. (1). For a linear strain harden-

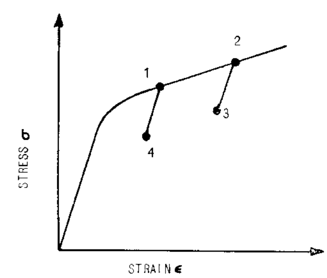

Fig. 2 Uniaxial strain curve with elastic unloading.

ing material the result is

$$\Delta u_c = \frac{1}{2E}[\sigma_{xQ}^2 + \sigma_{yQ}^2 - 2\upsilon\sigma_{xQ}\sigma_{yQ} + 2(1+\upsilon)\tau_{xyQ}^2 - \sigma_{x,Q-1}^2 - \sigma_{y,Q-1}^2 + 2\upsilon\sigma_{x,Q-1}\sigma_{y,Q-1} - 2(1+\upsilon)\tau_{xy,Q-1}^2] + \left(\sum_{q=1}^{Q-1}\Delta\epsilon_x^{(q)P}\right)(\sigma_{xQ} - \sigma_{x,Q-1}) + \left(\sum_{q=1}^{Q-1}\Delta\epsilon_y^{(q)P}\right)(\sigma_{yQ} - \sigma_{yQ-1}) + \left(\sum_{q=1}^{Q-1}\Delta\gamma_{xy}^{(q)P}\right)(\tau_{xyQ} - \tau_{xy,Q-1}) + \sum_{q=1}^{Q}(\alpha\Delta T)_q(\sigma_{xQ} + \sigma_{yQ}) - \sum_{q=1}^{Q}(\alpha\Delta T)_q(\sigma_{x,Q-1} + \sigma_{y,Q-1}) + \frac{1-m}{2mE}(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})^2 \quad (14)$$

The incremental complementary energy approach can be implemented in a discrete element context as follows. Let the structure be subdivided into K regions or zones of volume, V_k. The total incremental complementary energy $\Delta\Pi_c$ may be expressed as

$$\Delta\Pi_c = \sum_{k=1}^{K}\left\{\int_{V_k}\Delta u_c dV_k\right\} - \int_{S_2}(\bar{u}\Delta X + \bar{v}\Delta Y)dS \quad (15)$$

where the second term on the right hand side introduces the influence of displacement boundary conditions $(\bar{u},\bar{v})$. The symbol S_2 denotes that portion of the boundary where displacements are prescribed.

The static admissibility requirement in the interior of the structural system is satisfied in two parts; within each region, V_k, and between neighboring regions. In each region, V_k, statically admissible stresses may be represented in terms of a finite number of unknowns. This is accomplished by expressing the stresses in terms of a stress function, $\phi^{(k)}(x,y)$, and then approximating the stress function as the sum of products of one-dimensional Hermite interpolation polynomials (see Ref. 6) and undetermined coefficients. For the rectangular element, shown in Fig. 3, the assumed form of the stress function is

$$\phi = \sum_{i=1}^{2}\sum_{j=1}^{2}\{\phi_{ij}P_i(x)P_j(y) + \phi_{xij}P_{xi}(x)P_j(y) + \phi_{yij}P_i(x)P_{yj}(y) + \phi_{xxij}P_{xxi}(x)P_j(y) + \phi_{yyij}P_i(x)P_{yyj}(y) + \phi_{xyij}P_{xi}(x)P_{yj}(x) + \phi_{xxyij}P_{xxi}(x)P_{yj}(y) + \phi_{xyyij}P_{xi}(x)P_{yyj}(y) + \phi_{xxyyij}P_{xxi}(x)P_{yyj}(y)\} \quad (16)$$

where

$$\begin{aligned} P_1(x) &= 1 - 10(x/a)^3 + 15(x/a)^4 - 6(x/a)^5 \\ P_2(x) &= 10(x/a)^3 - 15(x/a)^4 + 6(x/a)^5 \\ P_{x1}(x) &= a[(x/a) - 6(x/a)^3 + 8(x/a)^4 - 3(x/a)^5] \\ P_{x2}(x) &= a[-4(x/a)^3 + 7(x/a)^4 - 3(x/a)^5] \\ P_{xx1}(x) &= (a^2/2)[(x/a)^2 - 3(x/a)^3 + 3(x/a)^4 - (x/a)^5] \\ P_{xx2}(x) &= (a^2/2)[(x/a)^3 - 2(x/a)^4 + (x/a)^5] \end{aligned} \quad (17)$$

and a denotes the element length in the x direction. The polynomials P_j, P_{yj}, etc., are obtained from Eqs. (17) by replacing x by y and a by b, where b is the element length in the y direction.

The subscripted ϕ's are the values of the stress function or its derivatives at the corner designated by ij. For example, ϕ_{xxyy22} is $\partial^4\phi/\partial x^2\partial y^2$ at corners (2,2). This representation for the stress function involves 36 unknowns for each element.

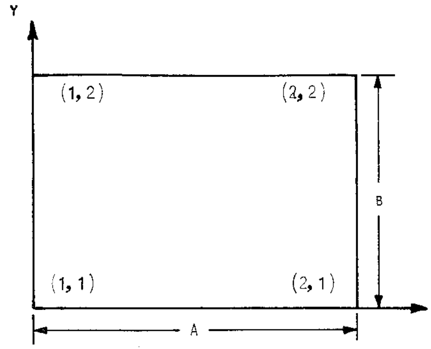

Fig. 3 Rectangular membrane element.

The stresses are related to the stress function by

$$\sigma_x = \partial^2\phi/\partial y^2,\ \sigma_y = \partial^2\phi/\partial x^2,\ \tau_{xy} = -\partial^2\phi/\partial x\partial y \quad (18a)$$

and therefore, assuming zero body forces, the equilibrium equations

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = 0,\ \partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y = 0 \quad (18b)$$

are satisfied identically within each region, V_k. Between neighboring regions statically admissible stresses must satisfy Newton's third law, which requires that the normal and shear stresses of neighboring regions be matched point for point along a common boundary. Constructing statically admissible stress states between neighboring regions is accomplished by equating the values of certain unknowns associated with one element to the appropriate unknowns associated with the neighboring element (see Appendix III of Ref. 7).

In cases where a physical insight into a problem leads to the expectation that the stresses are continuous throughout the structure, it is permissible to also continuously match the normal stresses that are parallel to the common boundaries. On external boundaries of the structure, prescribed force boundary conditions can be assigned by specifying the values of appropriate unknowns.

Executing the integration over the volume $\int_{V_k}\Delta u_c^{(k)}dV_k$ gives the contribution of kth region to the incremental complementary energy. The integration was performed in two parts. A closed form result was obtained for the terms which represent the elastic portion. The remaining terms were integrated numerically for the following two reasons: 1) a closed form solution was not available and 2) the elastic plastic interface can only be found numerically. Summing the integrals for the K regions and subtracting the incremental complementary work associated with the prescribed displacements gives the incremental complementary energy for the entire structure [see Eq. (15)]. Minimization of the incremental complementary energy with respect to the independent unknowns yields a solution for the Qth stress state. The minimization is accomplished using the technique described by Fletcher and Powell in Ref. 8. The Qth stress state is then treated as an initial stress state and the load is incremented upward. This process is continued until the desired load level is reached.

Rectangular Plate in Plane Stress

The incremental complementary energy approach was applied to a plane stress thermal problem (previously solved in Ref. 9). An isotropic material with linear strain hardening was assumed. The structure, shown in Fig. 4, is a rectangular plate. A total of five load increments were used to raise the temperature distribution to

$$\Delta T = 5.7\sigma_0 y^2/(\alpha E) \quad (19)$$

where σ_0 is the yield stress, α is the coefficient of linear

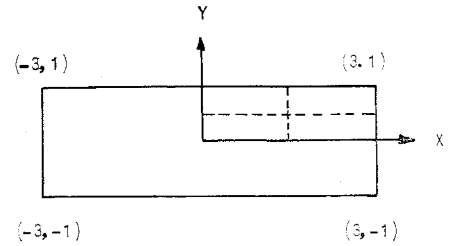

Fig. 4 Rectangular plate with coordinate system.

thermal expansion, and E is the modulus of elasticity. The five increments consisted of changing the numerical coefficient in Eq. (19) in the following sequence; 2.5, 3.5, 4.5, 5.5, 5.7. Using the prescribed temperature change given by Eq. (19), the incremental complementary energy can be expressed in terms of the nondimensional stresses σ_x/σ_0, σ_y/σ_0, and τ_{xy}/σ_0. Due to the absence of prescribed displacement boundary conditions, the stress function that minimizes $\Delta\Pi_c$ is independent of Poisson's ratio.‡ If $\Delta\Pi_c$ is expressed in terms of the nondimensional stresses, the only material property needed to obtain a stress solution is the strain hardening parameter m. This problem was worked in terms of the nondimensional stresses and a value of 0.1 was used for m.

Because of symmetry only one quadrant of the plate was considered. This plate quadrant was divided into four equal elements as shown by the dotted lines in Fig. 4. After force boundary conditions and interelement equilibrium conditions are satisfied and the additional information that the stresses are continuous in the plate is employed, the total number of independent degrees of freedom becomes 36 for the assemblage of four elements.

The following three aspects of the solution obtained were compared to the reference solution: 1) the final position of the plastic front, Fig. 5; 2) the normal stress $\sigma_x(0,y)$ at the cross section ($x = 0$) of the plate, Fig. 6; and 3) the mechanical strain (total strain minus thermal strain) at the cross section ($x = 0$) of the plate, Fig. 7. The curves in Figs. 6 and 7 represent the solution for a plate infinite in the x direction. The reference solution for the normal stress and mechanical strain at $x = 0$ falls on these curves. The circles represent results at specified values of y obtained by using the incremental complementary approach. These results were obtained using 36 independent degrees of freedom in the incremental complementary energy formulation, and required 6.5 minutes of machine running time for a Fortran IV program on the UNIVAC 1107 Computer at CWRU.

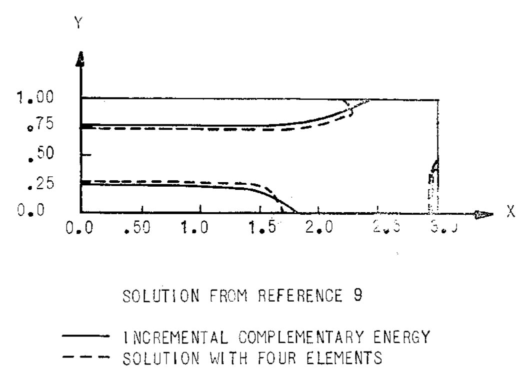

Fig. 5 Position of plastic front in fourth quadrant.

‡ The coefficient of Poisson's ratio appearing in the expression or $\Delta\Pi_c$, $[(\partial^2\phi/\partial y^2)(\partial^2\phi/\partial x^2) - (\partial^2\phi/\partial x\partial y)^2]$, identically satisfies he Euler differential equation for the first variation of $\Delta\Pi_c$.

Extensions to Include Orthotropic Materials that Behave Differently in Tension than Compression

Basically, the modifications in the formulation to include orthotropic materials that behave differently in tension and compression are in the stress-strain relations. The remainder of the formulation is unchanged.

The following description of the modifications is in terms of an axisymmetric plane strain problem. For illustration, only one stress-strain relation is needed:

$$\epsilon_r = \frac{\sigma_r}{E_r} - \mu_{r\theta}\frac{\sigma_\theta}{E_\theta} - \mu_{rz}\frac{\sigma_z}{E_z} + \sum_{q=1}^{Q} (\alpha_r \Delta T)_q + \sum_{q=1}^{Q-1} \Delta\epsilon_r^{P(q)} + \Delta\epsilon_r^{P(Q)} \quad (20)$$

where the first three terms in Eq. (20) represent the linear strain, the fourth term represents the thermal strain, up to and including the Qth load increment, the fifth term represents the accumulated plastic-strain due the previous $Q - 1$ load increments, and the last term represents the incremental plastic strain for the Qth load increment. The expression for the incremental strain, $\Delta\epsilon_r^{P(Q)}$, is

$$\Delta\epsilon_r^{P(Q)} = \tfrac{1}{2}(\Delta\bar{\epsilon}^P/\bar{\sigma})(\partial\bar{f}/\partial\sigma_r) \quad (21)$$

where

$$\Delta\bar{\epsilon}^P = (2)^{1/2}/3\{(F_r\Delta\epsilon_r^P - F_\theta\Delta\epsilon_\theta^P)^2 + (F_r\Delta\epsilon_r^P - F_z\Delta\epsilon_z^P)^2 + (F_\theta\Delta\epsilon_\theta^P - F_z\Delta\epsilon_z^P)^2\}^{1/2} \quad (22)$$

and

$$\bar{f} = \bar{\sigma}^2 = \left(\frac{\sigma_r}{F_r}\right)^2 + \left(\frac{\sigma_\theta}{F_\theta}\right)^2 + \left(\frac{\sigma_z}{F_z}\right)^2 - \frac{\sigma_r\sigma_\theta}{F_rF_\theta} - \frac{\sigma_r\sigma_z}{F_rF_z} - \frac{\sigma_\theta\sigma_z}{F_\theta F_z} \quad (23)$$

The different behavior in tension and compression is introduced by means of the F's in Eqs. (22) and (23) (see Ref. 10). For example, if $\sigma_r > 0$, $F_r = F_r^T$ otherwise $F_r = F_r^C$, where F_r^T and F_r^C denote the magnitudes of the stresses at the first change in slope of the uniaxial stress-strain curve in tension and compression respectively.

For an isotropic material, a uniaxial stress-strain curve in any direction can be used to determine an expression for $\Delta\bar{\epsilon}^P/\bar{\sigma}$. However, this is not always true for an orthotropic material because the stress-strain curves in the three principal material directions are different. To establish a representative and unique expression for $\Delta\bar{\epsilon}^P/\bar{\sigma}$ the following procedure

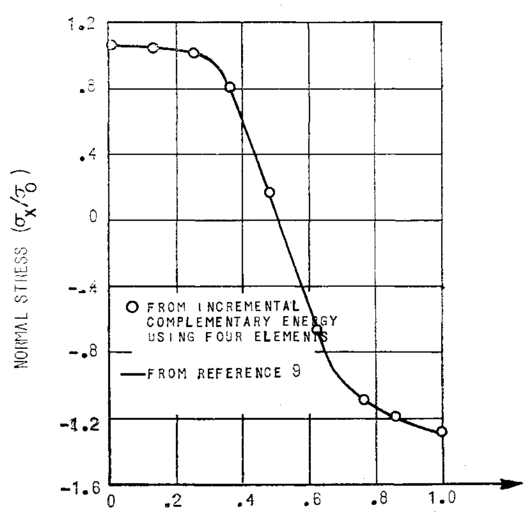

Fig. 6 Nondimensional stress at cross section $X = 0$.

was used. First a piecewise linear strain hardening curve was used to represent the uniaxial stress-strain in the stiffest material direction. Then piecewise linear strain hardening stress-strain curves for the "weaker" material directions were constructed from the requirement $\Delta\bar{\epsilon}^P/\bar{\sigma}$ stiffest direction = $\Delta\bar{\epsilon}^P/\bar{\sigma}$ "weaker" directions. In particular, this procedure was employed using stress-strain curves[11] for JTA material.§

For the axisymmetric problem, the radial equilibrium equation assuming zero body forces is

$$(\partial\sigma_r/\partial r) + [(\sigma_r - \sigma_\theta)/r] = 0 \tag{24}$$

The "stress function" $\phi(r)$, used to satisfy Eq. (24) is related to the stresses σ_r and σ_θ by the following relations:

$$\begin{aligned} \sigma_r &= \phi(r) \\ \sigma_\theta &= \sigma_r + r(\partial\sigma_r/\partial r) = \phi(r) + r[\partial\phi(r)/\partial r] \end{aligned} \tag{25}$$

where r is the conventional radius in the (r,θ) polar coordinate system. The discrete elements used here are hollow thick-walled right circular cylinders of thickness a.

The stress function associated with the kth element, $\phi^{(k)}(r)$, is approximated by the series

$$\phi(r) = \phi_1P_1(r) + \phi_2P_2(r) + \phi_{r1}P_{r1}(r) + \phi_{r2}P_{r2}(r) + \phi_{rr1}P_{rr1}(r) + \phi_{rr2}P_{rr2}(r) \tag{26}$$

where the P's are identical with the polynomials given in Eq. (17) with x replaced by r. The values of ϕ and its first two derivatives at the inner and outer surface of the element are treated as the unknowns.

Equilibrium between adjacent elements is satisfied by equating σ_r at the common boundaries of adjacent elements.

Applications

This analysis was applied to several composite cylinders[7] for which experimental results were available in Ref. 13. One application was to a copper core steel case composite with an area ratio of steel to copper of 0.5. The steel and copper were treated as isotropic linear strain hardening materials that behave the same in tension and compression. Figure 8 shows the experimental load-deflection curve and selected points on the predicted load-deflection curve ob-

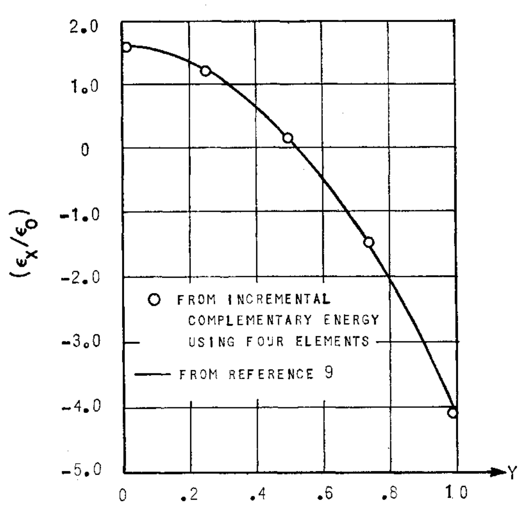

Fig. 7 Nondimensional mechanical strain (total strain minus thermal strain) at cross section $X = 0$.

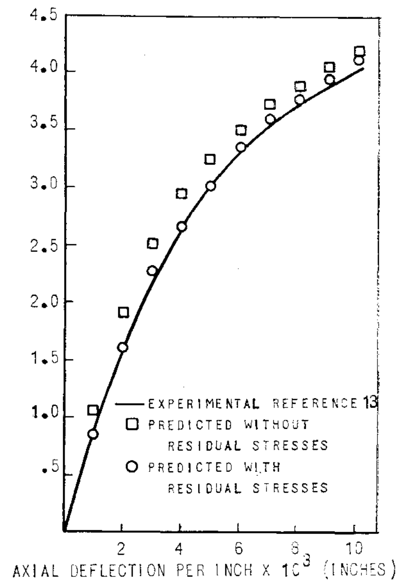

Fig. 8 Load-deflection curve for copper steel composite. Area ratio of steel to copper = 0.5. Area of composite = 0.0755 in.^{2}

tained by 1) assuming the composite is stress free when tested and 2) including an estimate of the residual stresses due to fabrication. The residual stresses were predicted by assuming a stress-free state existed at 550°F above room temperature. One increment was used to drop the temperature by 550°F while the composite cylinder was subjected to generalized plane strain so that the resultant axial force vanished. Room temperature material properties were used. It is observed from Fig. 8 that including an estimate of residual stresses gives better correlation with the experimental curve.

Another application of the incremental complementary energy approach was to predict the strains on the outer surface of several JTA cylinders subjected to axisymmetric pressure loadings.[7] The JTA material was treated as a transversely isotropic material that behaves differently in tension than compression. A comparison of the measured[11] and predicted tangential strains on the outer surface of one of JTA cylinders is shown in Fig. 9.

Summary of Results

The incremental complementary energy approach to stress analysis including material nonlinearity has been successfully applied to plane stress and generalized plane strain problems. A procedure for treating orthotropic nonlinear materials that behave differently in tension than compression has been suggested. Structures were modeled as an assemblage of discrete elements. Load incrementation was used. Stress

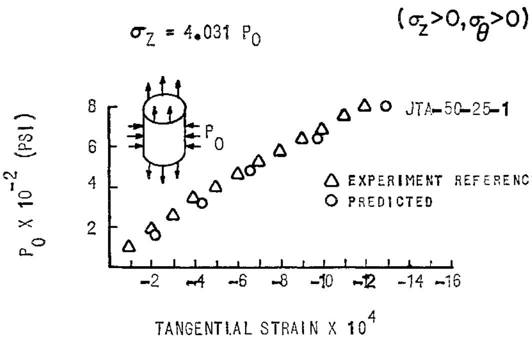

Fig. 9 Tangential strains on outer surface of JTA cylinder.

§ JTA material is transversely isotropic and behaves differently in tension and compression. By weight, JTA is 48% carbon, 43% zirconium diboride, and 9% silicon. The JTA material is fabricated by a single-step hot forming process.[12]

solutions were obtained by minimization of the incremental complementary energy associated with each load increment. The availability of powerful unconstrained minimization techniques[8] and a rational scaling procedure[14] contributed significantly to the algorithmic efficiency of the method.

Two types of discrete elements were successfully employed: a rectangular membrane element and an axisymmetric cylindrical element. The rectangular element was a two-dimensional element capable of describing plane stress. The cylindrical element was a one-dimensional element capable of describing an axisymmetric triaxial stress state (σ_r, σ_θ, and σ_z). Equilibrium conditions were satisfied exactly within each discrete element by introducing a stress function for each element. Satisfying the interelement equilibrium conditions exactly was facilitated by using Hermite interpolation polynomials[6] to approximate the stress functions. In the case of the plate in plane stress, the hyperosculatory polynomials can exactly represent a prescribed boundary stress which is a cubic function of the edge coordinate. More complex prescribed boundary stresses can be approached by increasing the number of discrete elements. The formulation also handles prescribed displacement boundary conditions, approximately, as natural boundary conditions. Thus, mixed boundary conditions can be dealt with. Intrinsic to the method is the fact that compatibility conditions are satisfied approximately.

There are certain features, inherent to the method, that facilitate the formulation and solution of the problem. The incremental complementary energy formulation does not require development of a compatibility equation in terms of the stress function. The boundary conditions on the portion of the boundary where displacements are prescribed do not have to be derived. Material nonlinearities are easily included in the formulation without altering the basic approach. Solutions are obtained by minimization algorithms which provide an alternative source of iterative procedures for dealing with multiaxial nonlinear material behavior in structural analysis.

Appendix A: Theoretical Verification for the Incremental Complementary Energy Approach

The principle of stationary complementary energy states that among all stress states that satisfy equilibrium and stress boundary conditions, that which renders the complementary energy stationary satisfies the compatibility conditions and the displacement boundary conditions and is the true stress state. It is well known that for linear stress-strain relations, linear strain displacement equations, and small rotations, the stationary value of the complementary energy is also a minimum. In this appendix it is shown that the stationary "point" for the incremental complementary energy is unique, minimizes the incremental complementary energy, and is the solution to the incremental problem.

It is also shown that if a region is specified to unload during a given load increment (here "unloading" is defined by the path 1-4 in Fig. 2, where the value of the equivalent stress at the end of the previous load increment is at point 1 in the figure), then the stationary point of the incremental complementary energy solves the problem for that load increment, minimizes the incremental complementary energy, and is unique.

The theoretical verification for the incremental complementary energy method is divided into three steps. Steps 1 and 2 pertain to "loading" and step 3 treats a specified case of unloading.

Step 1

It is shown that the stationary value of the incremental complementary energy provides the compatibility equation and the displacement boundary conditions. Hence, since equilibrium and stress-boundary conditions are satisfied a priori, the stationary point corresponds to a solution to the incremental problem.

Step 2

It is shown that the incremental complementary energy is a strictly convex function of the stresses. A convex set of allowable stresses is defined. Since the minimum of a strictly convex function over a convex set is unique,[15] the incremental complementary energy has only one minimum which is a unique stationary point.

Step 3

A specialized type of "unloading" is defined. For this case it is shown that a minimum of the incremental complementary energy is unique and corresponds to a unique solution to the incremental problem.

In the following it is assumed that the strains have unique and finitely defined first and second partial derivatives with respect to the stresses.

Step 1: Equivalence between a Stationary Point of $\Delta\pi_c$ and the Solution for a Given Load Increment

In order to show that a stationary value of $\Delta\pi_c$ is a solution to the problem, the first variation of $\Delta\pi_c$ is set to zero. This condition produces the compatibility equation and the displacement boundary conditions.

For the purpose of illustration, a plane stress case for a material that behaves the same in tension as compression is considered. However, a similar proof exists for the extension of the method to include different behavior in tension and compression. The expression for $\Delta\pi_c$ can be written as

$$\Delta\pi_c = \int_A\int \Delta u_c(\sigma_x,\sigma_z,\tau_{xy})dA - \int_{S_2} (\bar{u}_n\Delta\sigma_n + \bar{u}_t\Delta\tau_{nt})dS \quad \text{(A1)}$$

where the only requirements on Δu_c are

$$\partial\Delta u_c/\partial\sigma_x = \epsilon_x,\ \partial\Delta u_c/\partial\sigma_y = \epsilon_y,\ \partial\Delta u_c/\partial\tau_{xy} = \gamma_{xy} \quad \text{(A2)}¶$$

$\bar{u}_n$ and $\bar{u}_t$ are prescribed normal and tangential displacements on the boundary, and S_2 is the portion of the boundary where displacements are prescribed.

Expressing the stresses in terms of the Airy stress function,

$$\sigma_x = \phi_{yy},\ \sigma_y = \phi_{xx},\ \tau_{xy} = -\phi_{xy} \quad \text{(A3)}$$

and substituting these relations into Eq. (A1) gives

$$\Delta\pi_c = \int_A\int \Delta u_c(\phi_{yy},\phi_{yy}, -\phi_{xy})dA - \int_{S_2} (\bar{u}_n\Delta\phi_{tt} - \bar{u}_t\Delta\phi_{nt})dS \quad \text{(A4)}$$

The first variation of $\Delta\pi_c$ for rectangular regions can be written as

$$\delta\Delta\pi_c = \int_A\int \left\{\frac{\partial\Delta u_c}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\Delta u_c}{\partial\phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\Delta u_c}{\partial\phi_y}\right) + \frac{\partial^2}{\partial\chi^2}\left(\frac{\partial\Delta u_c}{\partial\phi_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial\Delta u_c}{\partial\phi_{yy}}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial\Delta u_c}{\partial\phi_{xy}}\right)\right\}\delta\phi dA + \oint\left[\frac{\partial\Delta u_c}{\partial\phi_x} - \frac{\partial}{\partial x}\left(\frac{\partial\Delta u_c}{\partial\phi_{xy}}\right)\right]\delta\phi dy - \oint\left[\frac{\partial\Delta u_c}{\partial\phi_y} - \frac{\partial}{\partial y}\left(\frac{\partial\Delta u_c}{\partial\phi_{xy}}\right)\right]\delta\phi dx + \oint\delta\phi_x\frac{\partial\Delta u_c}{\partial\phi_{xx}}dy - \oint\delta\phi_y\frac{\partial\Delta u_c}{\partial\phi_{yy}}dx + \oint\delta\phi\frac{\partial}{\partial x}\left(\frac{\partial\Delta u_c}{\partial\phi_{xy}}\right)dx + \oint\delta\phi_y\frac{\partial\Delta u_c}{\partial\phi_{xy}}dy - \int_{S_2}(\bar{u}_n\delta\phi_{tt} - \bar{u}_t\delta\phi_{nt})dS \quad \text{(A5)}$$

¶ These requirements are satisfied because $d\Delta u_c = \epsilon_x d\sigma_x + \epsilon_y\sigma_y + \gamma_{xy}d\tau_{xy}$.

where the $\oint$ denotes integration on the boundary in a counter-clockwise direction. Setting $\delta\Delta\pi_c$ equal to zero gives the following Euler equation:

$$\frac{\partial \Delta u_c}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \Delta u_c}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \Delta u_c}{\partial \phi_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Delta u_c}{\partial \phi_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial \Delta u_c}{\partial \phi_{yz}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial \Delta u_c}{\partial \phi_{xy}}\right) = 0 \text{ in the area, } A \quad \text{(A6)}$$

Because Δu_c is not a function of ϕ, ϕ_x, and ϕ_y,

$$\frac{\partial \Delta u_c}{\partial \phi} = 0, \frac{\partial \Delta u_c}{\partial \phi_x} = 0, \frac{\partial \Delta u_c}{\partial \phi_y} = 0 \quad \text{(A7)}$$

Combining Eqs. (A2) and (A3) gives

$$\begin{aligned} \partial \Delta u_c/\partial \phi_{xx} &= \partial \Delta u_c/\partial \sigma_y = \epsilon_y = v_y \\ \partial \Delta u_c/\partial \phi_{yy} &= \partial \Delta u_c/\partial \sigma_x = \epsilon_x = u_x \quad \text{(A8)} \\ \partial \Delta u_c/\partial \phi_{xy} &= -\partial \Delta u_c/\partial \tau_{xy} = -\gamma_{xy} = -u_y - v_x \end{aligned}$$

Substituting Eqs. (A7) and (A8) into Eq. (A6) gives the well-known compatibility equation

$$(\partial^2/\partial x^2)(\epsilon_y) + (\partial^2/\partial y^2)(\epsilon_x) + (\partial^2/\partial x \partial y)(-\gamma_{xy}) = 0 \quad \text{(A9)}$$

The line integral terms in Eq. (A5) contain the quantities $\delta\phi$, $\delta\phi_x$, $\delta\phi_y$. In order to obtain the natural boundary conditions in terms of the stresses, these terms are integrated by parts to obtain the quantities $\delta\phi_{xx}$, $\delta\phi_{yy}$, and $\delta\phi_{xy}$. Carrying out the integration by parts and replacing the area integral with Eq. (A9), $\delta\Delta\pi_c$ can be written as

$$\delta\Delta\pi_c = \int_A\int \left\{\frac{\partial^2}{\partial x^2}\epsilon_y + \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\right\}\delta\phi dA + \oint \delta\phi_{yy} u dy - \oint \delta\phi_{xx} v dx - \oint \delta\phi_{xy} v dy + \oint \delta\phi_{xy} u dx - \int_{S_2} (\bar{u}_n \delta\phi_{tt} - \bar{u}_t \delta\phi_{nt}) dS \quad \text{(A10)}$$

The last term of Eq. (A10) can be expressed as

$$-\oint_{S_2}(\bar{u}\delta\phi_{yy} - \bar{v}\delta\phi_{xy})dy - \oint_{S_2}(-\bar{v}\delta\phi_{xx} + \bar{u}\delta\phi_{xy})dx \quad \text{(A11)}$$

Substituting Eq. (A11) into Eq. (A10), noting that $\delta\phi_{yy}$, $\delta\phi_{xy}$, and $\delta\phi_{xx}$, are zero except on the S_2 portion of the boundary, and regrouping terms gives

$$\delta\Delta\pi_c = \int_A\int \left\{\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\right\}\delta\phi dA + \oint_{S_2}(u - \bar{u})\delta\phi_{yy} dy - \oint_{S_2}(v - \bar{v})\delta\phi_{xy} dy - \oint_{S_2}(v - \bar{v})\delta\phi_{xx} dx + \oint_{S_2}(u - \bar{u})\delta\phi_{xy} dx \quad \text{(A12)}$$

The last four terms represent the natural boundary conditions. Thus, setting $\delta\Delta\pi_c = 0$ yields the compatibility equation and the boundary conditions on the portion of the boundary where displacements are prescribed and, hence, solves the problem.

Step 2: Convex Properties of $\Delta\pi_c$

Since a minimum of $\Delta\pi_c$ is by definition a stationary point of $\Delta\pi_c$, a minimum solves the problem. It remains to be shown that $\Delta\pi_c$ has a unique minimum and that this minimum is the only stationary point of $\Delta\pi_c$. This is accomplished by observing that $\Delta\pi_c$ is strictly convex. A strictly convex function cannot have a local maximum or a saddle point because the "neighborhoods" of local maxima and saddle points are not convex.

The expression for $\Delta\pi_c$ can be written as

$$\Delta\pi_c = \int_A\int \left\{u_c{}^E + \sum_{q=1}^{Q}(\alpha_x \Delta T)_q \sigma_x + \sum_{q=1}^{Q}(\alpha_y \Delta T)_q \sigma_y + \sum_{q=1}^{Q-1}\Delta\epsilon_x{}^{(q)p}\sigma_x + \sum_{q=1}^{Q-1}\Delta\epsilon_x{}^{(q)p}\sigma_y + \sum_{q=1}^{Q-1}\Delta\gamma_{xy}{}^{(q)p}\tau_{xy} + k(\bar{\sigma} - \bar{\sigma}_{Q-1})^2\right\}dA - \int_{S_2}(\bar{u}_n \Delta\sigma_n + \bar{u}_t \Delta\sigma_t)dS + \text{constant terms} \quad \text{(A13)}$$

where k is a positive constant and $u_c{}^E$ is the elastic complementary energy density. Since $u_c{}^E$ is quadratic and positive definite, it is strictly convex.[16] The linear terms in σ_x, σ_y, and τ_{xy} are convex. (Linear terms are both convex and concave as are constants.)

It remains to be shown that the quantity $(\sigma_Q - \bar{\sigma}_{Q-1})^2$ is convex. This is done by first showing that $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})$ is convex. Using this information and the fact that $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})$ is always greater than or equal to zero, it is shown that $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})^2$ is convex. It can be shown that the quantity $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})$ is convex by examining its matrix of second partial derivatives. A quantity is convex if its matrix of second partial derivatives is positive semidefinite.[17] Since $\bar{\sigma}_{Q-1}$ is constant, the matrix of second partial derivatives is

$$\begin{bmatrix} \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_x{}^2} & \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_x \partial \sigma_y} & \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_x \partial \tau_{xy}} \\ \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_x \partial \sigma_y} & \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_y{}^2} & \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_y \partial \tau_{xy}} \\ \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_x \partial \tau_{xy}} & \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \sigma_y \partial \tau_{xy}} & \dfrac{\partial^2 \bar{\sigma}_Q}{\partial \tau_{xy}{}^2} \end{bmatrix}$$

Each of the diagonal terms is greater than or equal to zero, as is the "2 × 2 determinant" involving the (1,1), (1,2), (2,1), and (2,2) terms. The determinant of the 3 × 3 matrix is zero. Thus, $\bar{\sigma}_Q - \bar{\sigma}_{Q-1}$ is positive semidefinite and convex.

To show that $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})^2$ is convex, start with the following inequality that holds for a convex function[18]:

$$[\bar{\sigma}(c) - \bar{\sigma}_{Q-1}] \leq \alpha[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}] + (1 - \alpha)[\bar{\sigma}(b) - \bar{\sigma}_{Q-1}] \quad \text{(A14)}$$

where a and b are two stress states such that

$$\bar{\sigma}(a) \geq \bar{\sigma}_{Q-1} \text{ and } \bar{\sigma}(b) \geq \bar{\sigma}_{Q-1}$$

The stress state c equals $\alpha a + (1 - \alpha)b$ where $0 \leq \alpha \leq 1$. Since both sides of (A14) are non-negative, the inequality holds if both sides are squared. Doing this gives

$$[\bar{\sigma}(c) - \bar{\sigma}_{Q-1}]^2 \leq \alpha^2[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}]^2 + (1 - \alpha)^2 \times [\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]^2 + 2\alpha(1 - \alpha)[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}][\bar{\sigma}(b) - \bar{\sigma}_{Q-1}] \quad \text{(A15)}$$

Rearranging the following inequality

$$\{[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}] - [\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]\}^2 \geq 0$$

gives

$$[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}]^2 + [\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]^2 \geq 2[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}] \times [\bar{\sigma}(b) - \bar{\sigma}_{Q-1}] \quad \text{(A16)}$$

Replacing the product $2[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}][\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]$ in (A15) with $[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}]^2 + [\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]^2$ gives

$$[\bar{\sigma}(c) - \bar{\sigma}_{Q-1}]^2 \leq \alpha^2[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}]^2 + (1 - 2\alpha + \alpha^2) \times [\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]^2 + (\alpha - \alpha^2)[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}]^2 + (\alpha - \alpha^2)[\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]^2 \quad \text{(A17)}$$

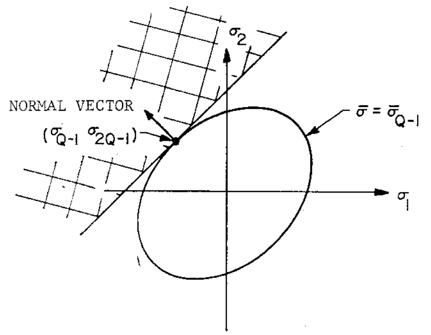

Fig. 10 Curve defining $\bar{\sigma} = \bar{\sigma}_{Q-1}$ in the σ_1, σ_2 plane.

Cancelling like terms with opposite signs gives

$$[\bar{\sigma}(c) - \bar{\sigma}_{Q-1}]^2 \leq \alpha[\bar{\sigma}(a) - \bar{\sigma}_{Q-1}]^2 + (1 - \alpha)[\bar{\sigma}(b) - \bar{\sigma}_{Q-1}]^2 \quad \text{(A18)}$$

which means that $(\bar{\sigma}_Q - \bar{\sigma}_{-1})^2$ is convex. Thus, $\Delta\pi_c$ is a strictly convex function because it is the sum of strictly convex and convex terms.

Showing that $\Delta\pi_c$ is a strictly convex function is not sufficient to guarantee a unique minimum. The set of stresses over which $\Delta\pi_c$ is minimized must be a convex set.[15] This requirement is met by defining loading to be such that $\bar{\sigma}_Q \geq \bar{\sigma}_{Q-1}$ and requiring the vector dot product of

$$(\sigma_{xQ},\sigma_{YQ},\tau_{xyQ}) \text{ and } (\partial\bar{\sigma}_{Q-1}/\partial\sigma_x,\partial\bar{\sigma}_{Q-1}/\partial\sigma_y,\partial\bar{\sigma}_{Q-1}/\partial\tau_{xy}) \text{ be } \geq 0$$

In terms of principal stresses, this definition of loading corresponds to the cross-hatched area of Fig. 10. With this definition of loading there is a unique minimum of $\Delta\pi_c$.

Step 3: Special Case of "Unloading"

The following pertains to the special case where it is known which points in the material will "unload."

For the type of unloading defined by the path 1-4 in Fig. 2, where the equivalent stress at the end of the previous load increment is given by the stress level at point 1 of the figure, it can be shown that the stationary point of $\Delta\pi_c$ solves the problem and that $\Delta\pi_c$ is strictly convex.

The correspondence between a stationary point and the solution for a load increment follows that given for loading with the term $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})^2$ omitted. The strict convexity property of $\Delta\pi_c$ can be shown because the elastic complementary energy density is positive definite and the remaining terms are linear. (The quantity $(\bar{\sigma}_Q - \bar{\sigma}_{Q-1})^2$ is not present for unloading.) In order to obtain a unique minimum for $\Delta\pi_c$, the stresses over which it is minimized must be a convex set. This is accomplished if the allowable stress states are on the other side of the line bordering the cross-hatch area of Fig. 10. However, the requirement $\bar{\sigma}_Q \leq \bar{\sigma}_{Q-1}$ further restricts the allowable stress states to be in the "oval" bounded by $\bar{\sigma} = \bar{\sigma}_Q$ of Fig. 10.

References

[1] Westergaard, H. M., "On the Method of Complementary Energy," *American Society of Civil Engineers Proceedings*, Vol. 67, Feb. 1941, pp. 199–227.

[2] Wilder, T. W., III, "An Analysis of Statically Indeterminate Trusses Having Members Stresses Beyond the Proportional Limit," TN 2886, Feb. 1953, NACA.

[3] Libove, C., "Complementary Energy Method for Finite Deformations," *Proceedings of the American Society of Civil Engineers, Journal of the Engineering Mechanics Division*, Vol. 90, No. EM6, Dec. 1964, pp. 49–71.

[4] DeVeubeke, B. F., "Displacement and Equilibrium Models in the Finite Element Method," *Stress Analysis*, edited by O. C. Zienkiewicz and G. S. Hollister, Wiley, London, 1965, Chap. 9.

[5] Kamel, H. A. and Sack, R. L., "Computational Stability in Nonlinear Discrete Structural Systems," *AIAA/ASME 10th Structures, Structural Dynamics and Materials Conference*, AIAA, New York, 1969, pp. 114–124.

[6] Bogner, F. K., Fox, R. L., and Schmit, L. A., "The Generation of Interelement-Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas," *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Wright-Patterson Air Force Base, Oct. 1965, pp. 397–443.

[7] Rybicki, E. F., "An Incremental Complementary Energy Method of Nonlinear Stress Analysis," Doctoral thesis, 1968, Case Western Reserve Univ., Cleveland, Ohio.

[8] Fletcher, R. and Powell, M. J. D., "A Rapidly Convergent Descent Method for Minimization," *Computer Journal*, Vol. 6, 1963, pp. 163–168.

[9] Mendelson, A., "Solutions of Some Plane, Thermal and Crack Problems for Strain Hardening Materials," Doctoral thesis, 1965, Case Institute of Technology, Cleveland, Ohio.

[10] Norris, Charles B., "Strength of Orthotropic Materials Subjected to Combined Stresses," Rept. 1816, May 1962, Forest Products Lab., Madison, Wis.

[11] Weng, T. L., private communication, July 1967, Union Carbide Corp., Parma Technical Center, Parma, Ohio.

[12] Tanzilli, R. A. and Ramke, W. G., "Evolution of Graphite Composites in Re-entry Environments," *Proceedings of the AIAA/ASME Seventh Structures and Materials Conference*, AIAA, New York, 1966, p. 290.

[13] Ebert, L. J., Hamilton, C. H., and Hecker, S. S., "Development of Design Criteria for Composite Materials," AFML-TR-67-95, April 1967, Air Force Materials Lab, Dayton, Ohio.

[14] Fox, R. L. and Stanton, E. L., "Developments in Structural Analysis by Direct Energy Minimization," *AIAA Journal*, Vol. 6, No. 6, June 1968, pp. 1036–1042.

[15] Hadley, G., *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, Mass., 1964, pp. 90–91.

[16] Hadley, G., *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, Mass., 1964, p. 85.

[17] Collatz, L., *Functional Analysis and Numerical Mathematics*, translated by H. Oser, Academic Press, New York, 1966, pp. 335–336.

[18] Hadley, G., *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, Mass., 1964, p. 83.